Orthogonal-Cyclic Latin Squares of Orders 9, 15, 21, and 25.

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ABSTRACT

Using a Commodore 64 to enumerate various possibilities, cyclic latin squares of orders 9, 15, 21, and 25 were investigated. The particular cyclic squares studied were generated by ci+j where $c = 1,2,\cdots,n-1$, $i = 0,1,\cdots,n-1$, and $j = 0,1,2,\cdots,n-1$. The computer program indicated which pairs of the n-1 taken 2 at a time pairs were orthogonal. One interesting fact was that several POLS(21,3) sets were obtained, which is a new result. For the other values of n, no more than a POLS(n,p-1) set, where p is the smallest prime factor of n, resulted from this method of generating cyclic latin squares.

1. INTRODUCTION

It is well know that the smallest prime integer in n minus one pairwise orthogonal latin squares can be formed by cyclical permutations of rows of a latin square of odd order. We denote this set as POLS(n,p_1-1), where $n = p_1p_2p_3\cdots$ for $p_1 \le p_2 \le p_3\cdots$. It is not known if there can be more than p_1-1 pairwise orthogonal latin squares except for some n, e.g. n = 15 where a POLS(15,3) set was found by Hedayat (1971) and a POLS(15,4) set was found by Schellenberg *et at.* (1978). This investigation uses a

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particular method of permuting rows of cyclic latin squares of order $n = 9$, 15, 21, and 25 to ascertain if the chosen method produces more than a pair, i.e. 3-1, for $n = 9$, 15, and 21, and more than four, i.e. 5-1, for $n = 25$. The method is to use the row permutations obtained from $ci+j$, modulo n, for $i,j = 0,1,\cdots,n-1$ and $c = 2,3,\cdots,n-1$. When $c = 1$ we obtain the original square of the form

	Column						
Row	1	2	3	4	$\cdots$	n-1	n
1	0	1	2	3		n-2	n-1
2	1	2	3	4		n-1	0
3	2	3	4	5		0	1
4	3	4	5	6		1	2
$\vdots$							
n	n-1	0	1	2		n-3	n-2

when $c = 2$, modulo n, we obtain the following square:

	Column						
Row	1	2	3	4	$\cdots$	n-1	n
1	0	1	2	3		n-2	n-1
2	2	3	4	5		0	1
3	4	5	6	7		2	3
4	6	7	8	9		4	5
$\vdots$							
n	n-2	n-1	0	1		n-4	n-3

For n = 9 the two squares are:

0	1	2	3	4	5	6	7	8
1	2	3	4	5	6	7	8	0
2	3	4	5	6	7	8	0	1
3	4	5	6	7	8	0	1	2
4	5	6	7	8	0	1	2	3
5	6	7	8	0	1	2	3	4
6	7	8	0	1	2	3	4	5
7	8	0	1	2	3	4	5	6
8	0	1	2	3	4	5	6	7

0	1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	0	1
4	5	6	7	8	0	1	2	3
6	7	8	0	1	2	3	4	5
8	0	1	2	3	4	5	6	7
1	2	3	4	5	6	7	8	0
3	4	5	6	7	8	0	1	2
5	6	7	8	0	1	2	3	4
7	8	0	1	2	3	4	5	6

The remaining squares for c = 3,···,n-1 are obtained by a similar process.

A computer program (see appendix) was written to compare all possible pairs of such squares for orthogonality. Note that this method does not give all possible permutations of the last n-1 rows of the original square. This is the subject of another study.

This investigation was carried out for n = 9, 15, 21, and 25. The results are given below.

2. n = 9

For n = 9, there are n-1 taken 2 at a time = 8 taken 2 at a time = 28 pairs to be studied. In the program, c was the value used to generate the first square and d=c+1,c+2,···,n-1 was used to generate the second square. Each pair was compared for orthogonality. Note that it was necessary to compare only one element, e.g. zero, of square one with the elements of square two superimposed upon square one. This is true because of the cyclic nature of the two squares. To illustrate consider the two squares for n = 9 given in the previous section. The element zero in the first square appears with zero in row one, with one in row two, with 2 in the row

3, with 3 in row 4, with 4 in row 5, with 5 in row 6, with 6 in row 7, with 7 in row 8, and with 8 in the last row. Zero in square one occurred once with each of the elements in square two and therefore the two squares are orthogonal (designated by O in Table 1.) Of the 28 pairs in Table 1, only nine pairs (1,2), (1,5), (1,8), (2,4), (2,7), (4,5), (4,8), (5,7), and (7,8), were orthogonal. 13 (marked X) of the squares formed were not even latin squares. These squares had rows 0, 3, and 6 of square one each repeated three times; the remaining 6 squares were latin squares but were not orthogonal (marked N) to the other member of the pair in Table 1. Thus, there were 9 Os, 6 Ns, and 13 Xs in the 28 pairs.

Table 1. 28 pairs of latin squares of order 9 compared for orthogonality.

	d =						
c =	2	3	4	5	6	7	8
1	O	X	N	O	X	N	O
2		X	O	N	X	O	N
3			X	X	X	X	X
4				O	X	N	O
5						O	N
6						X	X
7							O

O denotes orthogonality

N denotes nonorthogonality

X denotes the square is not a latin square

When c = 1, latin squares were not formed for d = 3 or 6. If we delete these two members, then we are considering 8-2 taken 2 at a time = 15 latin squares which is what was obtained.

Only pairs of orthogonal latin squares were obtained by this method. No set of three pairwise orthogonal latin squares resulted from the method. It still is not know if cyclic permutations of the rows of a cyclic latin will produce more than a pair of orthogonal latin squares for n = 9.

3. n = 15

Following the procedure described above 91 squares were formed of which 28 were latin squares (Table 2). Comparing pairs of the 28 latin squares for orthogonality, we note that 12 pairs were orthogonal and 16 pairs were not. Note that **no** trio of latin squares was found which were pairwise orthogonal. This method then did not produce a trio of pairwise latin squares as was found by Hedayat (1971).

We note that for every c and d for which 3 or 5 are factors, the square formed is not a latin square. For c = 3, 6, 9, 12, and 15 = 0, mod 15, in CI+J(I,J = 0,1,···,14), we obtain the sequence 0,3,6,9,12,0,3,6,9, 12,0,3,6, 9,12, in the first column of the square, this means that rows 0,3,6,9, and 12 from the original latin square (i.e. c = 1) are each repeated three times. For c = 5 or 10 in CI+J, we obtain the sequence 0,5, 10,0,5,10,0,5,10,0,5,10,0,5,10 which gives a square for which rows 0,5, and 10 of the original square are each repeated five times.

Table 2. 91 pairs of latin squares of order 15 compared for orthogonality

c =	d = 2	3	4	5	6	7	8	9	10	11	12	13	14
1	O	X	N	X	X	N	O	X	X	N	X	N	O
2		X	O	X	X	N	N	X	X	N	X	O	N
3			X	X	X	X	X	X	X	X	X	X	X
4				X	X	N	O	X	X	O	X	N	N
5					X	X	X	X	X	X	X	X	X
6						X	X	X	X	X	X	X	X
7							O	X	X	O	X	N	O
8								X	X	N	X	N	N
9									X	X	X	X	X
10										X	X	X	X
11	O: 12										X	O	N
12	N: 16											X	X
13	X: 63			Total: 91									O

c =	d = (3 or 5 are not factors of c or d) 2	4	7	8	11	13	14
1	O	N	N	O	N	N	O
2		O	N	N	N	O	N
4			N	O	O	N	N
7				O	O	N	O
8	O: 12				N	N	N
11	N: 16					O	N
13	Total: 28						O

O denotes orthogonality

N denotes nonorthogonality

X denotes not a latin square

4. n = 21

Repeating the procedure described above, we note that latin squares are not obtained when c in ci+j has 3 or 7 as a factor. Hence, these numbers were not considered. Thus, instead of studying 20 taken 2 at a time = 190 pairs of squares, the problem is reduced to considering 12 taken 2 at a time = 66 pairs. These are given in Table 3. Of these 66, there are 31 pairs which are orthogonal and 35 which are not. Of the 35, 7 have the pattern of nonorthogonality where elements 0,7, and 14 of square two each occur seven times with the element 0 of square one; the other 28 have the nonorthogonality pattern where elements 0,3,6,9,12,15, and 18 of square two occur three times each with element 0 of square one.

Two interesting features that occurred in Table 3. were the occurrence of triples and the relatively large number of triples of pairwise orthogonal latin squares. For example, the pairs (1,2), (1,10), and (2,10) from a triple; other triples are (1,5), (5,10), and (1,10); (1,10), (1,11), and (10,11); and (1,20), (1,10), and (10,20). The reason that this is interesting is that only pairs of orthogonal squares are obtained by Kronecher producting of orthogonal squares or orders 3 and 7. This appears to be a new result.

One item of note is that for all squares which are not latin squares, an obtained latin square is orthogonal to it in the sense that the element zero of the latin square appears once with each element of the nonlatin square.

Table 3. 66 pairs of cyclic latin squares of order 21 compared for orthogonality, c and d are not divisible by 3 or 7.

	d =										
c =	2	4	5	8	10	11	13	16	17	19	20
1	O	N	O	N	O	O	M	M	O	M	O
2		O	M	M	O	M	O	N	M	O	M
4			O	O	M	N	M	M	O	M	O
5				M	O	M	O	O	M	N	M
8					O	M	O	O	M	O	M
10						O	M	M	N	M	O
11							O	O	M	O	M
13	O:31							M	O	M	N
16	M:28								O	M	O
17	N:7									O	M
19	Total:66										O

O denotes orthogonality

M denotes nonorthogonality for which the element zero of the first squares occurs 3 times each with elements 0,3,6,9,12,15, 18 of square two.

N denotes nonorthogonality for which the element zero of the first square occurs 7 times each with elements 0,7,14 of the square two.

5. n = 25

For n = 25, there are four values of c for which cyclic latin squares are not formed from ci+j, i,j = 0,1,···,24 and c = 1,2,···,24. These values are c = 5,10,15, and 20. Therefore, there are 24-4 taken 2 at a time = 210 pairs of latin squares which need to be compared for non-orthogonality. As may be noted in Table 4, 40 (4+8+12+16) pairs were non-orthogonal and 170 pairs were orthogonal. Note the diagonal appearance of the letter N in Table 4.

The facts that sets of three pairwise orthogonal latin squares when $n = 21$ and that such a relatively large number of orthogonal pairs, 170, were obtained would appear to indicate that sets of more than four pairwise latin squares would result. This, however, is **not** the case. There are no POLS(25,5) sets in the table. The strategic location of the letter N in Table 4 prevents this.

Table 4. 210 pairs of cyclic latin squares of order 25 compared for orthogonality, c and d not factors of 25

	d =																		
c =	2	3	4	6	7	8	9	11	12	13	14	16	17	18	19	21	22	23	24
1	O	O	O	N	O	O	O	N	O	O	O	N	O	O	O	N	O	O	O
2		O	O	O	N	O	O	O	N	O	O	O	N	O	O	O	N	O	O
3			O	O	O	N	O	O	O	N	O	O	O	N	O	O	O	N	O
4				O	O	O	N	O	O	O	N	O	O	O	N	O	O	O	N
6					O	O	O	N	O	O	O	N	O	O	O	N	O	O	O
7						O	O	O	N	O	O	O	N	O	O	O	N	O	O
8							O	O	O	N	O	O	O	N	O	O	O	N	O
9								O	O	O	N	O	O	O	N	O	O	O	N
11									O	O	O	N	O	O	O	N	O	O	O
12										O	O	O	N	O	O	O	N	O	O
13											O	O	O	N	O	O	O	N	O
14												O	O	O	N	O	O	O	N
16													O	O	O	N	O	O	O
17														O	O	O	N	O	O
18															O	O	O	N	O
19																O	O	O	N
21																	O	O	O
22																		O	O
23																			O

O denotes orthogonality

N denotes nonorthogonality for which the element zero in first square, c values, occurs 5 times each with elements 0,5,10,15, 20 of square two.

6. Comments and Discussion

Let $n = ab$ where a and b are prime numbers. Then the number of values of $c = 1,2,\cdots,n-1$ which do not form latin squares is $(a-1) + (b-1)$. For example, let $n = 35$, then when $c = 7,14,21$, or 28, or when $c = 5,10,15,20,$ 25, or 30, the squares formed by $ci+j$ are not latin squares. These are $(5-1) + (7-1) = 10$ such values. For $n = p^k$, there are $(n/p^{k-1}) - 1$ values of c which do not form latin squares. The counting becomes more difficult when $n = abc,\cdots$, for a,b,c not prime numbers; this is because of duplication of values of c. For example, when $n = 45$, $c = 15$ is a factor of 3 and of 5. The number of values for which c does not form a latin square of order 45 is $(45/5 - 1) + (45/3 - 1) - 2 = 20$.

From the results obtained for $n = 21$, it would be interesting to study the situations for $n = 27,33,35,39,45,51$, and 55. The Commodore 64 used to study the cases $n = 9,15,21$,and 25 appears to be too slow to study the above numbers. It could be speeded up by not printing A(I,J) and B(I,J). Likewise, the computer program in the appendix may need to be extended to produce Tables 1 and 4 directly. This would lessen the time required at the computer. It is estimated that 10-15 hours, depending upon how much the printer was used, were required for this study. More time than this would be required for $n = 27$, and the time increases considerably as n increases.

7. References Cited

Hedayat, A. (1971) A set of three mutually orthogonal latin squares of order 15. *Technometrics* **13(3)**: 696-698.

Schellenberg, P. J., van Rees, G. H. J., and Vanstone, S. A. (1978) Four pairwise orthogonal latin squares of order 15. *Ars Combinatoria* **6**: 141-150.

Appendix

Program is written in BASIC for Commodore 64 to generate cyclic latin squares using the generator CII+JJ. For each value of the pair C, D, two squares are generated and compared for orthogonality.

```
5 REM POLS(N,2)8/85
10 INPUT "VALUE FOR N=";N
20 DIM A(N,N), B(N,N)
30 FOR C = 1 TO N-2
40 FOR D = C+1 TO N-1
50 FOR I = 1 TO N
60 FOR J = 1 TO N
70 II = I-1 : JJ = J-1
80 A(I,J) = INT((C*II+JJ) - N*INT((C*II+JJ)/N))
90 B(I,J) = INT((D*II+JJ) - N*INT((D*II+JJ)/N))
100 NEXT J
110 NEXT I
120 :
130 CLOSE 3
140 OPEN3,4 : CMD3
150 PRINT "N=";N,"C=";C,"D=";D
160 PRINT : PRINT : PRINT : PRINT
170 FOR I = 1 TO N
180 PRINT "A";
190 FOR J = 1 TO N
200 PRINT A(I,J);
210 NEXT J
220 PRINT
230 PRINT "B";
240 FOR J = 1 TO N
250 PRINT B(I,J);
260 NEXT J
270 PRINT : PRINT
280 NEXT I
290 FOR I = 1 TO N
300 FOR J = 1 TO N
310 IF A(I,J) = 0 THEN PRINT I,J,B(I,J)
320 NEXT J
330 NEXT I
340 NEXT D
350 NEXT C
360 IF C < N-1 GOTO 30
370 IF C > N-2 THEN END
380 PRINT#3 : CLOSE 3

READY.
```

An example of the output for n = 4, the smallest integer producing more than one pair of squares, is given below:

```
N= 4              C= 1              D= 2

A 0  1  2  3
B 0  1  2  3

A 1  2  3  0
B 2  3  0  1

A 2  3  0  1
B 0  1  2  3

A 3  0  1  2
B 2  3  0  1

 1                1                0
 2                4                1
 3                3                2
 4                2                3

N= 4              C= 1              D= 3


A 0  1  2  3
B 0  1  2  3

A 1  2  3  0
B 3  0  1  2

A 2  3  0  1
B 2  3  0  1

A 3  0  1  2
B 1  2  3  0

 1                1                0
 2                4                2
 3                3                0
 4                2                2

N= 4              C= 2              D= 3
```

```
A 0  1  2  3
B 0  1  2  3

A 2  3  0  1
B 3  0  1  2

A 0  1  2  3
B 2  3  0  1

A 2  3  0  1
B 1  2  3  0

 1            1            0
 2            3            1
 3            1            2
 4            3            3

READY.
```

Note that square B for C = 2, D = 3 above is superimposed upon square A that is

$$B(I,J) = \begin{array}{|cccc|} \hline 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ \hline \end{array}$$

, a latin square,

is superimposed upon square

$$A(I,J) = \begin{array}{|cccc|} \hline 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ \hline \end{array}$$

, which is not

a latin square. The three columns immediately below the superimposed pair of square refer to row number where element 0 occurs (see program statement 310), the column number where element 0 occurred, and the element of B which occurred with zero in the I,J position, respectively. All elements $0,1,\cdots,N-1$ must each appear once in this last column or else the condition for orthogonality is not satisfied. Note that all elements do occur above for C = 2 and D = 3 **but** square A(I,J) is **not** a latin square. Therefore, this is not a POLS(4,2) set.

Some program notes occur below:

Note 1: If steps 100 and 110 is written as a single step, i.e. 100 NEXT J,I and likewise for steps 320 and 330, and replace steps 340 and 350 by 340 NEXT D,C, the program executes on the Commodore 64. Since this type of short cut depends upon the hardware, the individual steps were included in the program.

Note 2: If step 130 is omitted, the program executes for C = 1 and then shows a device error. Step 130 solved this problem.

Note 3: To have the results appear on the computer screen instead of on a hardcopy, simply omit steps 120, 130, and 140.

Note 4: For most situations, it is redundant and confusing to print A(I,J). To simply print the rest, omit steps 180 and 200.

Note 5: Shortening 160 and or 270 to fewer PRINT statements produced drastic changes in format. Likewise, omitting the semicolons in statement 200 and/or 250 also changes the above format considerably.